

Hierarchies of Manakov–Santini Type by Means of Rota–Baxter and Other Identities

Błażej M. SZABLIKOWSKI

Faculty of Physics, Adam Mickiewicz University, Umultowska 85, 61-614 Poznań, Poland

E-mail: bszablik@amu.edu.pl

Received January 11, 2016, in final form February 22, 2016; Published online February 27, 2016

<http://dx.doi.org/10.3842/SIGMA.2016.022>

Abstract. The Lax–Sato approach to the hierarchies of Manakov–Santini type is formalized in order to extend it to a more general class of integrable systems. For this purpose some linear operators are introduced, which must satisfy some integrability conditions, one of them is the Rota–Baxter identity. The theory is illustrated by means of the algebra of Laurent series, the related hierarchies are classified and examples, also new, of Manakov–Santini type systems are constructed, including those that are related to the dispersionless modified Kadomtsev–Petviashvili equation and so called dispersionless r -th systems.

Key words: Manakov–Santini hierarchy; Rota–Baxter identity; classical r -matrix formalism; generalized Lax hierarchies; integrable $(2 + 1)$ -dimensional systems

2010 Mathematics Subject Classification: 37K10; 37K30

1 Introduction

In recent years, one of the significant achievements in the theory of integrable systems was the construction of formal solutions of the Cauchy problems for a wide class of $(2 + 1)$ -dimensional dispersionless systems by means of the inverse scattering transform [14, 15, 16, 17, 18]. In this process one of the crucial steps was the introduction by Manakov and Santini of a two-field system that generalizes the dispersionless Kadomtsev–Petviashvili (KP) equation. The Manakov–Santini system possesses a non-Hamiltonian Lax pair and the construction of related hierarchy [15] within the Lax–Sato formalism [6, 8] unifies two original approaches based on different underlying structures: first by Takasaki and Takebe [33, 34] and the second one by Martínez Alonso and Shabat [21, 22, 23]. The Manakov–Santini hierarchy and its generalizations were further studied in several works, see for instance [6, 7, 8, 9, 10, 25].

The aim of this work is an extension of the Lax–Sato formalism of Manakov–Santini hierarchy to a more general class of integrable systems, in particular such as the dispersionless modified KP equation or the so-called r -th systems [2, 3, 4]. Influenced by the papers [7, 8, 9, 10] we generalize the Lax–Sato formalism of Manakov–Santini hierarchy by means of the Lax hierarchy (2.1), where two linear operators \mathcal{P} and \mathcal{R} are introduced. In Theorem 1 we find the conditions, on the operators \mathcal{P} and \mathcal{R} , for the mutual commutativity of equations from the hierarchy (2.1). One of the conditions turns out to be the well-known Rota–Baxter identity [1, 26, 27]. The general source of the relations in Theorem 1 is explained in Section 3. In fact, these relations are directly connected with those that are used in the work [31] for the construction of Frobenius manifolds in the cotangent bundles. In Section 4 we illustrate the above construction by means of the algebra of Laurent series. The related hierarchies (2.1) are classified and there are presented examples of integrable systems of Manakov–Santini type, including new ones.

2 Generalized hierarchy

Let \mathbb{A} be a commutative associative unital algebra¹. We define the generalized Lax hierarchy of evolution equations by

$$\Psi_{t_n} = A_n \Psi_x - B_n \partial \Psi, \quad \Psi = \begin{pmatrix} L \\ M \end{pmatrix}, \quad n \in \mathbb{N}, \quad (2.1)$$

where $L, M \in \mathbb{A}$ are Lax and Orlov operators (functions), respectively. The independent variables are evolution parameters (times) t_n and the spatial variable x . We assume that ∂ is some (auxiliary) derivation in the algebra \mathbb{A} invariant with respect to all t_n and x . The hierarchy is generated by the functions

$$A_n := \mathcal{P}(J^{-1} \partial X_n) \quad \text{and} \quad B_n := \mathcal{R}(J^{-1}(X_n)_x), \quad X_n := L^n, \quad (2.2)$$

where \mathcal{P} and \mathcal{R} are some linear maps $\mathbb{A} \rightarrow \mathbb{A}$ and

$$J := \{L, M\} \equiv \partial L M_x - L_x \partial M. \quad (2.3)$$

We assume that the endomorphisms \mathcal{P} and \mathcal{R} are invariant with respect to times t_n and the spatial variable x , that is \mathcal{P} and \mathcal{R} commute with derivatives related to t_n and x .

In all the following proofs we will skip most of straightforward computations, however we will exhibit all the crucial intermediate steps.

Proposition 1. *The evolution equations from the hierarchy (2.1) pairwise commute if the following pair of zero-curvature type equations is satisfied by the generating functions A_n and B_n :*

$$(A_n)_{t_m} - (A_m)_{t_n} + \langle A_n, A_m \rangle_x + B_m \partial A_n - B_n \partial A_m = 0 \quad (2.4a)$$

and

$$(B_n)_{t_m} - (B_m)_{t_n} + A_n (B_m)_x - A_m (B_n)_x - \langle B_n, B_m \rangle_\partial = 0, \quad (2.4b)$$

where

$$\langle a, b \rangle_x := ab_x - ba_x, \quad \langle a, b \rangle_\partial := a \partial b - b \partial a.$$

Proof. The commuting of the respective flows means that $(\Psi_{t_n})_{t_m} = (\Psi_{t_m})_{t_n}$. Comparing the coefficients of both sides with respect to the independent variables Ψ_x and $\partial \Psi$ we obtain the required pair of zero-curvature conditions. \blacksquare

Theorem 1. *The following set of conditions on the endomorphisms \mathcal{P} and \mathcal{R} :*

$$\mathcal{P}(\mathcal{P}(a)b + a\mathcal{P}(b)) - \mathcal{P}(a)\mathcal{P}(b) = \kappa_1 \mathcal{P}(ab), \quad (2.5a)$$

$$\mathcal{P}(\mathcal{R}(a)\partial b + a\partial \mathcal{P}(b)) - \mathcal{R}(a)\partial \mathcal{P}(b) = \kappa_1 \mathcal{P}(a\partial b), \quad (2.5b)$$

$$\mathcal{R}(\mathcal{P}(a)b + a\mathcal{R}(b)) - \mathcal{P}(a)\mathcal{R}(b) = \kappa_2 \mathcal{R}(ab), \quad (2.5c)$$

$$\mathcal{R}(\mathcal{R}(a)\partial b + a\partial \mathcal{R}(b)) - \mathcal{R}(a)\partial \mathcal{R}(b) = \kappa_2 \mathcal{R}(a\partial b), \quad (2.5d)$$

where $a, b \in \mathbb{A}$ and $\kappa_1, \kappa_2 \in \mathbb{C}$, is sufficient for the zero-curvature equations (2.4) to be identically fulfilled by the generating functions A_n and B_n . This means that the above equations are sufficient conditions for mutual commutation of the flows from the hierarchy (2.1).

¹We assume, for simplicity, that all structures in this work are defined over the field of complex numbers.

Proof. Let's define

$$a_n := J^{-1}\partial X_n \quad \text{and} \quad b_n := J^{-1}(X_n)_x,$$

so that $A_n = \mathcal{P}(a_n)$ and $B_n = \mathcal{R}(b_n)$. Using (2.1) one finds that

$$(X_n)_{t_m} = A_m(X_n)_x - B_m\partial X_n$$

and

$$J_{t_n} = (A_n J)_x - \partial(B_n J) \quad \Rightarrow \quad (J^{-1})_{t_n} = \langle A_n, J^{-1} \rangle_x - \langle B_n, J^{-1} \rangle_\partial.$$

Hence,

$$(a_n)_{t_m} = \langle A_m, a_n \rangle_x + b_n \partial A_m - B_m \partial a_n, \quad (b_n)_{t_m} = A_m(b_n)_x - a_n(B_m)_x - \langle B_m, b_n \rangle_\partial.$$

Now substituting $(A_n)_{t_m} \equiv \mathcal{R}((a_n)_{t_m})$ and $(B_n)_{t_m} \equiv \mathcal{R}((a_n)_{t_m})$ into the pair of zero-curvature equations (2.4) and taking advantage of the relations (2.5) one finds that

$$\begin{aligned} (A_n)_{t_m} + A_n(A_m)_x + B_m\partial A_n - n &\leftrightarrow m \\ &= \mathcal{P}(A_m(a_n)_x + a_m(A_n)_x) - A_m(A_n)_x - \mathcal{P}(B_m\partial a_n + b_m\partial A_n) + B_m\partial A_n - n \leftrightarrow m \\ &= \kappa_1 \mathcal{P}(a_m(a_n)_x - a_n(a_m)_x - b_m\partial a_n + b_n\partial a_m) \\ &= \kappa_1 \mathcal{P}(J^{-1}\partial(J^{-1}\{X_m, X_n\})) = 0 \end{aligned}$$

and

$$\begin{aligned} (B_n)_{t_m} - A_m(B_n)_x - B_n\partial B_m - n &\leftrightarrow m \\ &= \mathcal{R}(A_m(b_n)_x + a_m(B_n)_x) - A_m(B_n)_x - \mathcal{R}(B_m\partial b_n + b_m\partial B_n) + B_m\partial B_n - n \leftrightarrow m \\ &= \kappa_2 \mathcal{R}(a_m(b_n)_x - a_n(b_m)_x - b_m\partial b_n + b_n\partial b_m) \\ &= \kappa_2 \mathcal{R}(J^{-1}(J^{-1}\{X_m, X_n\})_x) = 0. \end{aligned}$$

In the above $n \leftrightarrow m$ stands for all the remaining terms arising from the permutation of the indices m and n in the preceding terms. Since $\{X_m, X_n\} = 0$ we see that the zero-curvature equations (2.4) are indeed satisfied identically. \blacksquare

Lemma 1. Assume that the endomorphisms \mathcal{P} and \mathcal{R} satisfy the following relation

$$\mathcal{P}(\partial a) = \partial \mathcal{R}(a) \tag{2.6}$$

for arbitrary $a \in \mathbb{A}$. Then, under the constraint

$$J \equiv \{L, M\} = L^s, \tag{2.7}$$

where s is some fixed integer, the generalized hierarchy (2.1) reduces to the standard hierarchy of the form

$$L_{t_n} = \frac{n}{n-s} \{\mathcal{R}(L^{n-s}), L\}, \tag{2.8}$$

where the Poisson bracket is defined through the formula (2.3), that is $\{, \} := \partial \wedge \partial_x$.

On the other hand, if we assume that L is invariant with respect to times t_n and the variable x , that is $L_{t_n} = L_x = 0$, then the hierarchy reduces to

$$M_{t_n} = n \mathcal{P}(M_x^{-1} L^{n-1}) M_x. \tag{2.9}$$

Proof. Taking advantage of the constraint (2.7) and the relation (2.6) we have

$$A_n = \mathcal{P}(J^{-1}\partial X_n) = \frac{n}{n-r}\mathcal{P}(\partial L^{n-r}) = \frac{n}{n-r}\partial\mathcal{R}(L^{n-r})$$

and

$$B_n = \mathcal{R}(J^{-1}(X_n)_x) = \frac{n}{n-r}\mathcal{R}((L^{n-r})_x) = \frac{n}{n-r}(\mathcal{R}(L^{n-r}))_x.$$

The last relation follows from the assumption that \mathcal{R} commutes with ∂_x . As result the hierarchy (2.1) reduces into

$$\Psi_{t_n} = \frac{n}{n-r}[\partial\mathcal{R}(L^{n-r})\Psi_x - (\mathcal{R}(L^{n-r}))_x\partial\Psi] \equiv \frac{n}{n-r}\{\mathcal{R}(L^{n-r}), \Psi\},$$

where $\Psi = (L, M)^T$. The first equation on L coincides with (2.8). The second equation gives evolution of M consistent with the constraint (2.7).

If $L_{t_n} = L_x = 0$, then $B_n = 0$ and $J = \partial LM_x$. Hence

$$A_n = \mathcal{P}(J^{-1}\partial X_n) = n\mathcal{P}(M_x^{-1}L^{n-1}).$$

In this case, the first equation in (2.1) for $\Psi = L$ is satisfied identically and the second equation for $\Psi = M$ takes the form (2.9). ■

3 Rota–Baxter and other identities

Consider some algebra (\mathbb{A}, \cdot) . The Rota–Baxter identity [1, 26, 27] for some linear operator $\mathcal{P}: \mathbb{A} \rightarrow \mathbb{A}$ has the form

$$\mathcal{P}(\mathcal{P}(a) \cdot b + a \cdot \mathcal{P}(b)) - \mathcal{P}(a) \cdot \mathcal{P}(b) = \kappa \mathcal{P}(a \cdot b), \quad (3.1)$$

where $a, b, c \in \mathbb{A}$ and κ is some fixed scalar weight. Alternatively we can write the identity as

$$\ell(\ell(a) \cdot b + a \cdot \ell(b)) - \ell(a) \cdot \ell(b) = \frac{1}{4}\kappa^2 a \cdot b,$$

where $\ell := \mathcal{P} - \frac{1}{2}\kappa$. For an operator \mathcal{P} satisfying (3.1) there is always associated operator $\mathcal{P}' := \kappa - \mathcal{P}$, which satisfies the identity (3.1) for the same weight κ .

There is a source of simple solutions to the Rota–Baxter identity (3.1). Assume that the algebra \mathbb{A} can be decomposed into direct sum of subalgebras, that is

$$\mathbb{A} = \mathbb{A}_+ \oplus \mathbb{A}_-, \quad \mathbb{A}_\pm \cdot \mathbb{A}_\pm \subset \mathbb{A}_\pm, \quad \mathbb{A}_+ \cap \mathbb{A}_- = \{0\}.$$

Then, the projections P_+ and P_- on the subalgebras \mathbb{A}_+ and \mathbb{A}_- satisfy the identity (3.1) for the weight $\kappa = 1$. Notice that $P_+ + P_- = 1$.

The main feature of the Rota–Baxter identity is that in the case of associative algebra \mathbb{A} the identity (3.1) is a sufficient condition for associativity of another multiplication in \mathbb{A} given by

$$a *_{\mathcal{P}} b := \mathcal{P}(a) \cdot b + a \cdot \mathcal{P}(b) - \kappa a \cdot b \equiv \ell(a) \cdot b + a \cdot \ell(b).$$

For more information on the Rota–Baxter algebras see [12, 13].

The special case of the Rota–Baxter identity (3.1), for a Lie algebra $(\mathbb{A}, [\cdot, \cdot])$, is the modified Yang–Baxter equation

$$\mathcal{R}([\mathcal{R}(a), b] + [a, \mathcal{R}(b)]) - [\mathcal{R}(a), \mathcal{R}(b)] = \kappa \mathcal{R}([a, b]). \quad (3.2)$$

The equation (3.2) is a sufficient condition for the bracket

$$[a, b]_{\mathcal{R}} := [\mathcal{R}(a), b] + [a, \mathcal{R}(b)] - \kappa[a, b]$$

to define second Lie bracket in \mathbb{A} , in this case the linear map R is called the classical r -matrix. The more standard convention is to consider the endomorphism $r := \mathcal{R} - \frac{1}{2}\kappa$ instead of \mathcal{R} . The classical r -matrix formalism [28, 29] is known to be very useful in the construction of very broad classes of integrable systems, see also the survey [5] and references therein.

Let's now consider commutative and associative algebra (\mathbb{A}, \cdot) . We define Poisson bracket in \mathbb{A} be means of two commuting derivations $\partial, \partial_x \in \text{Der } \mathbb{A}$:

$$\{a, b\} := \partial a \cdot \partial_x b - \partial_x a \cdot \partial b. \quad (3.3)$$

Here, the derivation ∂_x is a counterpart of the derivative with respect to a spatial variable x and the derivation ∂ is a counterpart of the derivative with respect to some auxiliary variable. The following identity on the endomorphism \mathcal{R} turns out to be important:

$$\mathcal{R}(\mathcal{R}(a)\partial b) + \mathcal{R}(a\partial\mathcal{R}(b)) - \mathcal{R}(a)\partial\mathcal{R}(b) = \kappa\mathcal{R}(a\partial b), \quad (3.4)$$

where κ is some constant. Analogically as before, $R' = \kappa - R$ solves (3.4) for the same weight κ .

Proposition 2. *Let us assume that the endomorphism \mathcal{R} commutes with the derivation ∂_x , that is $\mathcal{R}\partial_x = \partial_x\mathcal{R}$. Then, the identity (3.4) is a sufficient condition for \mathcal{R} to satisfy the modified Yang–Baxter equation*

$$\mathcal{R}(\{\mathcal{R}(a), b\} + \{a, \mathcal{R}(b)\}) - \{\mathcal{R}(a), \mathcal{R}(b)\} = \kappa\mathcal{R}(\{a, b\}), \quad (3.5)$$

and so to be a classical r -matrix for \mathcal{R} with respect to the Poisson bracket (3.3). This means that

$$\{a, b\}_{\mathcal{R}} := \{\mathcal{R}(a), b\} + \{a, \mathcal{R}(b)\} - \kappa\{a, b\}$$

is a Lie bracket when the relation (3.4) holds.

The proof is straightforward expanding the formula (3.4) by means of (3.3). In fact, when \mathcal{R} also commutes with the derivation ∂ , the Rota–Baxter identity (3.1) is sufficient for \mathcal{R} to solve the equation (3.4). However, in general (3.1) is more restrictive than (3.4).

Proposition 3. *Assume that, on a commutative associative algebra (\mathbb{A}, \cdot) , there exists non-degenerate bilinear product $(\cdot, \cdot)_{\mathbb{A}}: \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{C}$, such that the Frobenius condition holds:*

$$(a \cdot b, c)_{\mathbb{A}} = (a, b \cdot c)_{\mathbb{A}}$$

and the product is invariant with respect to the derivation $\partial \in \text{Der } \mathbb{A}$:

$$(\partial a, b)_{\mathbb{A}} + (a, \partial b)_{\mathbb{A}} = 0.$$

Then, the Rota–Baxter identity (3.1) is equivalent to the ‘dual’ relation:

$$\mathcal{R}(\mathcal{P}(a)b + a\mathcal{R}(b)) - \mathcal{P}(a)\mathcal{R}(b) = \kappa\mathcal{R}(ab), \quad \mathcal{R} := \kappa - \mathcal{P}^*,$$

where $(\mathcal{P}^*a, b)_{\mathbb{A}} := (a, \mathcal{P}b)_{\mathbb{A}}$.

Let additionally assume that the following relation is valid:

$$\mathcal{P}(\partial a) = \partial\mathcal{R}(a). \quad (3.6)$$

Then, from the Rota–Baxter identity (3.1) we obtain the relation (3.4) and

$$\mathcal{P}(\mathcal{R}(a)\partial b + a\partial\mathcal{P}(b)) - \mathcal{R}(a)\partial\mathcal{P}(b) = \kappa\mathcal{P}(a\partial b).$$

For proof see Proposition 3.3 and Lemma 5.3 in [31]². In the work [31] it is shown that the Rota–Baxter identity (3.1) and the equation (3.4) can be significant in the construction of Frobenius manifolds on cotangent bundles inherent to the integrable systems of hydrodynamic type.

Summarizing, the Rota–Baxter identity (3.1) coincides with the condition (2.5a) from Theorem 1, the sufficient condition (3.4) on \mathcal{R} to satisfy the modified Yang–Baxter equation (3.5) coincides with (2.5d), and if the relation (3.6) or (2.6) and some additional natural assumptions are satisfied, then the remaining conditions (2.5b) and (2.5c) hold automatically.

4 Application to the algebra of Laurent series

Consider the algebra of Laurent polynomials $\mathbb{A} := \mathbb{C}[p, p^{-1}]$, it is commutative and associative. When necessary the algebra \mathbb{A} can be extended to the algebra of formal Laurent series at ∞ : $\mathbb{A}^\infty := \mathbb{C}((p^{-1}))$ or the algebra of formal Laurent series at 0: $\mathbb{A}^0 := \mathbb{C}((p))$.

Consider decomposition of \mathbb{A} in the form

$$\mathbb{A} = \mathbb{A}_{\geq l} \oplus \mathbb{A}_{< l}, \quad A_{\geq l} := p^l \mathbb{C}[p], \quad A_{< l} := p^{l-1} \mathbb{C}[p^{-1}].$$

The related projections are defined by

$$\left[\sum_i a_i p^i \right]_{\geq l} := \sum_{i \geq l} a_i p^i \quad \text{and} \quad \left[\sum_i a_i p^i \right]_{< l} := \sum_{i < l} a_i p^i.$$

The subsets $\mathbb{A}_{\geq l}$ and $\mathbb{A}_{< l}$ are subalgebras only for $l = 0$ or $l = 1$. As result, the projections on these subalgebras solve the Rota–Baxter identity (3.1) or (2.5a) with the weight $\kappa = 1$, that is for

$$\mathcal{P} = [\cdot]_{\geq l} \quad \text{or} \quad \mathcal{P} = [\cdot]_{< l} \quad \text{if} \quad l = 0, 1. \quad (4.1)$$

Remember that $[\cdot]_{\geq l} + [\cdot]_{< l} = 1$.

We will look now for solutions of the identity (3.4), where we take the derivative:

$$\partial := p^r \partial_p, \quad r \in \mathbb{Z}.$$

Then, projections

$$\mathcal{R} = [\cdot]_{\geq k-r} \quad \text{or} \quad \mathcal{R} = [\cdot]_{< k-r} \quad (4.2)$$

solve the identity (3.4) or (2.5d) with $\kappa = 1$ if

- 1) $r = 0$ and $k = 0$;
- 2) $r \in \mathbb{Z}$ and $k = 1$ or $k = 2$;
- 3) $r = 2$ and $k = 3$.

Notice that the above solutions coincide with the r -matrices from [3, 32] with respect to the Poisson bracket defined by

$$\{, \}_r := p^r \partial_p \wedge \partial_x. \quad (4.3)$$

Proposition 4.

- All combinations of the above operators \mathcal{P} and \mathcal{R} , (4.1) and (4.2), that satisfy the identities (2.5a) and (2.5c) also satisfy the remaining identities (2.5b) and (2.5d).

²Notice the misprint in formula (5.9) in [31], there is missing minus sign on the right-hand side of the equality.

- For

$$\mathcal{P} = [\cdot]_{\geq l}, \quad \mathcal{R} = [\cdot]_{\geq k-r} \quad \text{and} \quad \partial = p^r \partial_p \quad (4.4a)$$

or

$$\mathcal{P} = [\cdot]_{< l}, \quad \mathcal{R} = [\cdot]_{< k-r} \quad \text{and} \quad \partial = p^r \partial_p \quad (4.4b)$$

in addition to the identities (2.5) the constraint (2.6) is also satisfied if

- 1) $k = 0$: $l = 0$ and $r = 0$;
 - 2) $k = 1$: $l = 0$ and $r \in \mathbb{Z}$ or $l = 1$ and $r = 1$;
 - 3) $k = 2$: $l = 1$ and $r \in \mathbb{Z}$ or $l = 0$ and $r = 1$;
 - 4) $k = 3$: $l = 1$ and $r = 2$.
- The respective hierarchies (2.1) for the solutions (4.4) turn out to be independent of r . That is, for (4.4a) we get

$$\Psi_{t_n^\infty} = \left[\frac{(L^n)_p}{\{L, M\}_0} \right]_{\geq l} \Psi_x - \left[\frac{(L^n)_x}{\{L, M\}_0} \right]_{\geq k} \Psi_p, \quad (4.5a)$$

where $\Psi = (L, M)^T$ for the Lax function $L \in \mathbb{A}^\infty$ and M being associated Orlov function. For (4.4b) we get

$$\Psi_{t_n^0} = \left[\frac{(\tilde{L}^n)_p}{\{\tilde{L}, \tilde{M}\}_0} \right]_{< l} \Psi_x - \left[\frac{(\tilde{L}^n)_x}{\{\tilde{L}, \tilde{M}\}_0} \right]_{< k} \Psi_p, \quad (4.5b)$$

where $\Psi = (\tilde{L}, \tilde{M})^T$ for the Lax function $\tilde{L} \in \mathbb{A}^0$ and \tilde{M} the associated Orlov function.

- The solutions (4.4a) and (4.4b) as well as the Lax hierarchies (4.5a) and (4.5b) are mutually equivalent through the transformation:

$$p' = p^{-1} \quad \text{with} \quad l' = 1 - l, \quad k' = 3 - k, \quad r' = 2 - r. \quad (4.6)$$

Proof. The first two points are consequence of straightforward verification. To show the third point consider the solutions (4.4a) and let $L \in \mathbb{A}^\infty$. Then, the generating functions (2.2) are

$$A_n = \left[\frac{p^r (L^n)_p}{\{L, M\}_r} \right]_{\geq l} = \left[\frac{(L^n)_p}{\{L, M\}_0} \right]_{\geq l}$$

and

$$B_n = \left[\frac{(L^n)_x}{\{L, M\}_r} \right]_{\geq k-r} = p^{-r} \left[\frac{(L^n)_x}{\{L, M\}_0} \right]_{\geq k},$$

where the Poisson bracket is defined by (4.3). Hence, the related hierarchy (2.1) takes the form (4.5a), since the factors p^{-r} and p^r cancel out. For the solutions (4.4b) and $\tilde{L} \in \mathbb{A}^0$ the reasoning is similar. The last point follows from simple verification. ■

For fixed parameters k, r and l we can consider the Lax functions L and \tilde{L} to be analytic extensions of some function outside and inside, respectively, unit circle on the complex plane. Thus L and \tilde{L} are defined near ∞ and 0 , respectively, where they have poles. We can choose these poles to be simple. As result we can consider two families of related hierarchies together and postulate $\Psi = (L, M, \tilde{L}, \tilde{M})^T$ in (4.5). This is, inter alia, consequence of the fact that the solutions (4.4a) and (4.4b) are mutually associated through the decomposition of unity:

$$1 = [\cdot]_{\geq l} + [\cdot]_{< l} = [\cdot]_{\geq k-r} + [\cdot]_{< k-r}.$$

The compatibility equations (2.4) for the hierarchies (4.5) takes the following, independent of r , form:

$$(A_n^\lambda)_{t_m^\mu} - (A_m^\mu)_{t_n^\lambda} + \langle A_n^\lambda, A_m^\mu \rangle_x + B_m^\mu (A_n^\lambda)_p - B_n^\lambda (A_m^\mu)_p = 0 \quad (4.7a)$$

and

$$(B_n^\lambda)_{t_m^\mu} - (B_m^\mu)_{t_n^\lambda} + A_n^\lambda (B_m^\mu)_x - A_m^\mu (B_n^\lambda)_x - \langle B_n^\lambda, B_m^\mu \rangle_p = 0, \quad (4.7b)$$

where $m, n \in \mathbb{N}$; $\mu, \lambda = \infty, 0$ and $\langle f, g \rangle_\xi := fg_\xi - gf_\xi$ for $\xi = x, p$. The related generating functions are defined as

$$A_n^\infty := \left[\frac{(L^n)_p}{\{L, M\}_0} \right]_{\geq l}, \quad B_n^\infty := \left[\frac{(L^n)_x}{\{L, M\}_0} \right]_{\geq k},$$

and

$$A_n^0 := \left[\frac{(\tilde{L}^n)_p}{\{\tilde{L}, \tilde{M}\}_0} \right]_{< l}, \quad B_n^0 := \left[\frac{(\tilde{L}^n)_x}{\{\tilde{L}, \tilde{M}\}_0} \right]_{< k}.$$

Notice that if we treat Ψ as a common eigenfunction, then the pairs of equations from (4.5) give the Lax pairs for the respective systems from (4.7).

Let us consider reductions from Lemma 1. First for constraints of the type (2.7) we take

$$\{L, M\}_r = L^r \quad \text{and} \quad \{\tilde{L}, \tilde{M}\}_r = \tilde{L}^{2-r}. \quad (4.8)$$

We assume that they are consistent with respect to the appropriate choice of Lax and Orlov functions, see the forthcoming examples. Then, in accordance with equations (2.7) and (2.8) the hierarchies (4.5) reduce to

$$L_{t_n^\infty} = \frac{n}{n-r} \{[L^{n-r}]_{\geq k-r}, L\}_r \quad \text{and} \quad \tilde{L}_{t_n^0} = \frac{n}{n-2+r} \{[\tilde{L}^{n+2-r}]_{< k-r}, \tilde{L}\}_r.$$

So we obtain the standard Poisson hierarchies of dispersionless systems, see [3, 32] and references therein. For the next reduction from Lemma 1 we postulate that

$$L = p \quad \text{and} \quad \tilde{L} = p^{-1}.$$

Then, the hierarchies (4.5) reduce to

$$M_{t_n^\infty} = n[p^{n-1}M_x^{-1}]_{\geq l}M_x \quad \text{and} \quad \tilde{M}_{t_n^0} = n[p^{1-n}\tilde{M}_x^{-1}]_{< l}\tilde{M}_x.$$

Taking $G := M_x^{-1}$ and $\tilde{G} := \tilde{M}_x^{-1}$ the above hierarchies can be rewritten in the form

$$G_{t_n^\infty} = n\langle [p^{n-1}G]_{\geq l}, G \rangle_x \quad \text{and} \quad \tilde{G}_{t_n^0} = n\langle [p^{1-n}\tilde{G}]_{< l}, \tilde{G} \rangle_x.$$

These are the universal hierarchies considered in [21, 22, 23].

For appropriate choice of Lax functions and associated Orlov functions the Lax hierarchies (4.5), in principle, yield construction of $(1+1)$ -dimensional integrable infinite-field (chain) systems, while the compatibility conditions (4.7) provide finite-field systems that include $(2+1)$ -dimensional integrable equations of Manakov–Santini type, which are of our interest. Although it would be fairly easy, we will not consider in this work finite-field reductions of Lax hierarchies (4.5).

4.1 The case of $k = l = r = 0$

The Lax function [2, 33, 34] has the form

$$L = p + u(x)p^{-1} + u_2(x)p^{-2} + u_3(x)p^{-3} + \cdots \in \mathbb{A}^\infty$$

and the associated Orlov function [33, 34] is

$$M = M_0 + x + v(x)L^{-1} + v_2(x)L^{-2} + \cdots,$$

where M_0 is the part of M that explicitly depends on times t_n^∞ and commutes with L , that is $\{M_0, L\}_0 \equiv 0$. This means that the choice of M_0 does not influence the construction of related systems from (4.7). In this case, consistent Lax hierarchy for $\tilde{L} \in \mathbb{A}^0$ does not exist.

Then, we calculate

$$\{L, M\}_0 = 1 + v_x p^{-1} + ((v_2)_x - u)p^{-2} + \cdots$$

and the first generating functions

$$\begin{aligned} A_1^\infty &= 1, & A_2^\infty &= p - v_x, & A_3^\infty &= p^2 - v_x p + 2u + v_x^2 - (v_2)_x, \\ B_1^\infty &= 0, & B_2^\infty &= u_x, & B_3^\infty &= u_x p + (u_2)_x - u_x v_x. \end{aligned}$$

From the generalized zero-curvature equations (4.7) for $n = 1$, $m = 2$ and $\lambda = \mu = \infty$ we get

$$u_{t_1} = u_x, \quad v_{t_1} = v_x,$$

where $t_1 \equiv t_1^\infty$. For $n = 1$, $m = 3$ we get

$$(u_1)_{t_1} = (u_1)_x, \quad (v_2)_{t_1} = (v_2)_x.$$

Which in fact means that we can identify the time t_1 with x . First nontrivial equations are for $n = 2$ and $m = 3$. Let $t \equiv t_3$ and $y \equiv t_2$. Hence, we obtain two compatibility conditions:

$$(v_2)_x = u + v_y + v_x^2, \quad (u_2)_x = u_y + u_x v_x$$

and the celebrated Manakov–Santini system [15]

$$\begin{aligned} v_{xt} &= v_{yy} + v_x v_{xy} + u v_{xx} - v_y v_{xx}, \\ u_{xt} &= u_{yy} + u_x^2 + u_{xy} v_x + u u_{xx} - u_{xx} v_y. \end{aligned} \tag{4.9}$$

The related Lax pair which follows from (4.5a) is

$$\begin{aligned} \partial_y \Psi &= [(p - v_x) \partial_x - u_x \partial_p] \Psi, \\ \partial_t \Psi &= [(p^2 - v_x p + u - v_y) \partial_x - (u_x p + u_y) \partial_p] \Psi. \end{aligned}$$

The reduction given by the condition $\{L, M\}_0 = 1$ means that $v = 0$. Thus, for $v = 0$ from (4.9) we obtain the dispersionless KP equation

$$u_{xt} = u_{yy} + u_x^2 + u u_{xx}.$$

The second possible reduction is given by the constraint: $L = p$, from which it follows that $u = 0$ and the system (4.9) reduces to the Pavlov equation [24] (see also [11, 22, 30])

$$v_{xt} = v_{yy} + v_x v_{xy} - v_y v_{xx}.$$

4.2 The case: $k = 1$, $l = 0$ and $r \in \mathbb{Z}$

The Lax and associated Orlov functions [2, 19, 20, 34] for $p \rightarrow \infty$ are given by

$$L = p + u(x) + u_1(x)p^{-1} + u_2(x)p^{-2} + \dots \in \mathbb{A}^\infty$$

and

$$M = M_0 + x + (\partial_x^{-1}u(x) + w(x))L^{-1} + w_2(x)L^{-2} + \dots,$$

where as before $\{M_0, L\}_0 \equiv 0$ and we made some modification for convenience of further calculations. Then

$$\{L, M\}_0 = 1 + (u + w_x)p^{-1} + ((w_2)_x - u_1 - u^2 - uw_x)p^{-2} + \dots$$

and

$$\begin{aligned} A_1^\infty &= 1, & A_2^\infty &= p - w_x, & A_3^\infty &= p^2 + (u - w_x)p + u^2 + uw_x + w_x^2 + 2u_1 - (w_2)_x, \\ B_1^\infty &= 0, & B_2^\infty &= u_x p, & B_3^\infty &= u_x p^2 + (uu_x - u_x w_x + (u_1)_x)p. \end{aligned}$$

From the generalized zero-curvature equations (4.7) for $n = 1$, $m = 2$ and $\lambda = \mu = \infty$ we get

$$u_{t_1} = u_x, \quad w_{t_1} = w_x,$$

where $t_1 \equiv t_1^\infty$. For $n = 1$, $m = 3$ we get

$$(u_1)_{t_1} = (u_1)_x, \quad (w_2)_{t_1} = (w_2)_x.$$

Which means that we can identify the time t_1 with x . First nontrivial equations are for $n = 2$, $m = 3$. Let $t \equiv t_3^\infty$ and $y \equiv t_2^\infty$. After some simplifications, we obtain two compatibility conditions

$$(u_1)_x = u_y + u_x w_x, \quad (w_2)_{xx} = 2u_y + w_{xy} + 2uu_x + 3u_x w_x + 2uw_{xx} + 2w_x w_{xx}$$

and the new $(2 + 1)$ -dimensional integrable system

$$\begin{aligned} u_{xt} &= u_{yy} + u_x u_y + u_x^2 w_x + uu_{xy} + u_{xy} w_x + u_{xx} a, \\ w_{xt} &= uw_{xy} + u_y w_x + w_x w_{xy} + aw_{xx} - a_y, \end{aligned} \tag{4.10}$$

where

$$a_x = u_x w_x - w_{xy}.$$

The Lax pair for the system (4.10) is given by

$$\begin{aligned} \partial_y \Psi &= [(p - w_x)\partial_x - u_x p \partial_p] \Psi, \\ \partial_t \Psi &= [(p^2 + (u - w_x)p - w_y - \partial_x^{-1}uw_{xx})\partial_x - (u_x p^2 + (uu_x + u_y)p)\partial_p] \Psi. \end{aligned}$$

Consider first reduction (4.8) given by the condition

$$\{L, M\}_r = L^r, \quad r \in \mathbb{Z}, \tag{4.11}$$

which is consistent since the order at ∞ of both sides of the equality is the same. Hence,

$$\{L, M\}_0 = p^{-r} L^r = 1 + rup^{-1} + \left(ru_1 - \frac{1}{2}r(1-r)u^2 \right) p^{-2} + \dots,$$

and we obtain the constraint

$$w_x = (r - 1)u$$

by means of which (4.10) gives the r -th dispersionless modified KP equation [2, 4]:

$$u_t = \frac{1}{2}(r - 1)u^2u_x + ruu_y + \partial_x^{-1}u_{yy} + (1 - r)u_x\partial_x^{-1}u_y,$$

which for $r = 0$ gives the standard dispersionless modified KP equation. The second reduction is given by the constraint: $L = p$, from which it follows that $u = 0$ and in this case the system (4.10) reduces again to the Pavlov equation

$$w_{xt} = w_{yy} + w_xw_{xy} - w_yw_{xx}.$$

The Lax and the associated Orlov functions for $p \rightarrow 0$ are

$$\tilde{L} = v(x)p^{-1} + v_0(x) + v_1(x)p + v_2(x)p^2 + \cdots \in \mathbb{A}^0$$

and

$$\tilde{M} = \tilde{M}_0 + m(x) + m_1(x)\tilde{L}^{-1} + m_2(x)\tilde{L}^{-2} + \cdots,$$

where $\{\tilde{M}_0, \tilde{L}\}_0 \equiv 0$. We have

$$\{\tilde{L}, \tilde{M}\}_0 = -vm_xp^{-2} - (m_1)_xp^{-1} + \cdots$$

and

$$\begin{aligned} A_1^0 &= 0, & A_2^0 &= \frac{v}{m_x}p^{-1}, & A_3^0 &= \frac{v^2}{m_x}p^{-2} + \frac{2vv_0m_x - v(m_1)_x}{m_x^2}p^{-1}, \\ B_1^0 &= 0, & B_2^0 &= -\frac{v_x}{m_x}, & B_3^0 &= -\frac{vv_x}{m_x}p^{-1} + \left(\frac{(m_1)_x}{m_x^2} - \frac{2v_0}{m_x}\right)v_x - \frac{v(v_0)_x}{m_x}. \end{aligned}$$

From the generalized zero-curvature equations (4.7) for $n = 2$, $m = 3$ and $\lambda = \mu = 0$ we get the constraints

$$(v_0)_x = \frac{v_zm_x}{v}, \quad \left(\frac{(m_1)_x}{m_x}\right)_x = m_{xz} + \frac{v_z}{v}m_x,$$

where $z \equiv t_2^0$, and the system

$$\left(\frac{v}{m_x}\right)_\tau = -\left[\frac{v}{m_x}\partial_x^{-1}v\left(\frac{m_x}{v}\right)_z\right]_z, \quad \left(\frac{v_x}{m_x}\right)_\tau = v_{zz} - \left[\frac{v_x}{m_x}\partial_x^{-1}v\left(\frac{m_x}{v}\right)_z\right]_z, \quad (4.12)$$

where $\tau \equiv t_3^0$. The Lax pair for (4.12) is given by

$$\begin{aligned} \partial_\tau \Psi &= \left[\frac{v}{m_x}p^{-1}\partial_x + \frac{v_x}{m_x}\partial_p\right]\Psi, \\ \partial_z \Psi &= \left[\left(\frac{v^2}{m_x}p^{-2} - \frac{v}{m_x}\partial_x^{-1}v\left(\frac{m_x}{v}\right)_z\right)p^{-1}\right]\partial_x + \left(\frac{vv_x}{m_x}p^{-1} + v_z - \frac{v_x}{m_x}\partial_x^{-1}v\left(\frac{m_x}{v}\right)_z\right)\partial_p\right]\Psi. \end{aligned}$$

Consider first reduction given by the condition

$$\{\tilde{L}, \tilde{M}\}_r = \tilde{L}^{2-r}, \quad r \in \mathbb{Z}, \quad (4.13)$$

which is consistent since order at 0 of both sides is the same. Thus,

$$\{\tilde{L}, \tilde{M}\}_0 = p^{-r} \tilde{L}^{2-r} = v^{2-r} p^{-2} + (2-r)v^{1-r} v_0 p^{-1} + \dots$$

and we obtain the constraint

$$m_x = -v^{1-r},$$

by means of which (4.12) gives the r -th dispersionless Harry–Dym equation [2, 4]:

$$v_\tau = -v^{1-r} [v^r \partial_x^{-1} v^{-r} v_z]_z.$$

The second reduction is given by the constraint: $\tilde{L} = p^{-1}$, from which it follows that $v = 1$ and the system (4.12) reduces to [22, 30]

$$m_{x\tau} = m_x m_{zz} - m_{xz} m_z.$$

Now, we will consider the mixed case. For $\lambda = \infty$, $n = 1$ and $\mu = 0$ with $m = 2$ or $m = 3$ in (4.7) we obtain

$$v_{t_1} = v_x, \quad m_{t_1} = m_x,$$

which holds automatically as $t_1 \equiv x$. From the zero-curvature equations (4.7), for $\lambda = \infty$, $n = 2$ and $\mu = 0$, $m = 2$, we obtain the following compatibility equations

$$u_z = \frac{v_x}{m_x}, \quad w_{xz} = -v \frac{m_{xx}}{m_x^2}$$

and

$$\left(\frac{v}{m_x} \right)_y = \frac{v}{m_x} (u_x + w_{xx}) - w_x \left(\frac{v}{m_x} \right)_x, \quad \left(\frac{v_x}{m_x} \right)_y = \frac{(u_x v)_x}{m_x} - w_x \left(\frac{v_x}{m_x} \right)_x.$$

The above equations can be equivalently rewritten in the following form, for which we have the conditions

$$u_x = (\log v)_y - (\log v)_x \frac{m_y}{m_x}, \quad w_x = -\frac{m_y}{m_x}$$

and the system

$$m_{xx} v = m_x m_{yz} - m_y m_{xz}, \quad v_{xx} = (\log v)_{yz} m_x - (\log v)_{xz} m_y, \quad (4.14)$$

which is a system of Manakov–Santini type recently obtained in [7]. The related Lax pair is

$$\partial_y \Psi = [(p + w) \partial_x - u_x p \partial_p] \Psi, \quad \partial_z \Psi = \left[\frac{v}{m_x} p^{-1} \partial_x + \frac{v_x}{m_x} \partial_p \right] \Psi.$$

Consider now the reduction by means of (4.11) and (4.13), that is we have constraints in the form

$$w_x = (r-1)u, \quad m_x = -v^{1-r}, \quad r \in \mathbb{Z}.$$

Hence, from (4.14) we obtain the r -th dispersionless Toda system [4, 19]:

$$u_z + v^{r-1} v_x = 0, \quad v_y = u_x v + (1-r)u v_x,$$

which for $r = 1$ gives the Boyer–Finley equation being $(2+1)$ -dimensional version of dispersionless Toda equation. On the other hand from the reduction: $L = p$, $\tilde{L} = p^{-1}$ we have the constraints: $u = 0$, $v = 1$, from which we get the equation [22, 24]

$$m_{xx} = m_x m_{yz} - m_y m_{xz}.$$

The remaining not considered cases from Proposition 4 are equivalent to the above examples through the transformation (4.6) or there is no consistent hierarchies for them on the level of equations (4.5).

Acknowledgement

The very preliminary results related to this work were obtained in 2010 during the visit of the author to Institute of Mathematics in Academia Sinica in Taipei. The author wishes to express his thanks to Professor Jen-Hsu Chang for valuable discussions and Professor Jyh-Hao Lee for warm hospitality.

References

- [1] Baxter G., An analytic problem whose solution follows from a simple algebraic identity, *Pacific J. Math.* **10** (1960), 731–742.
- [2] Błaszak M., Classical R -matrices on Poisson algebras and related dispersionless systems, *Phys. Lett. A* **297** (2002), 191–195.
- [3] Błaszak M., Szablikowski B.M., Classical R -matrix theory of dispersionless systems. I. $(1 + 1)$ -dimensional theory, *J. Phys. A: Math. Gen.* **35** (2002), 10325–10344, [nlin.SI/0211008](#).
- [4] Błaszak M., Szablikowski B.M., Classical R -matrix theory of dispersionless systems. II. $(2 + 1)$ dimension theory, *J. Phys. A: Math. Gen.* **35** (2002), 10345–10364, [nlin.SI/0211018](#).
- [5] Błaszak M., Szablikowski B.M., Classical R -matrix theory for bi-Hamiltonian field systems, *J. Phys. A: Math. Theor.* **42** (2009), 404002, 35 pages, [arXiv:0902.1511](#).
- [6] Bogdanov L.V., On a class of multidimensional integrable hierarchies and their reductions, *Theoret. and Math. Phys.* **160** (2009), 887–893, [arXiv:0810.2397](#).
- [7] Bogdanov L.V., Non-Hamiltonian generalizations of the dispersionless 2DTL hierarchy, *J. Phys. A: Math. Theor.* **43** (2010), 434008, 8 pages, [arXiv:1003.0287](#).
- [8] Bogdanov L.V., On a class of reductions of the Manakov–Santini hierarchy connected with the interpolating system, *J. Phys. A: Math. Theor.* **43** (2010), 115206, 11 pages, [arXiv:0910.4004](#).
- [9] Bogdanov L.V., Chang J.-H., Chen Y.-T., Generalized dKP: Manakov–Santini hierarchy and its waterbag reduction, [arXiv:0810.0556](#).
- [10] Chang J.-H., Chen Y.-T., Hodograph solutions for the Manakov–Santini equation, *J. Math. Phys.* **51** (2010), 042701, 18 pages, [arXiv:0904.4595](#).
- [11] Dunajski M., A class of Einstein–Weyl spaces associated to an integrable system of hydrodynamic type, *J. Geom. Phys.* **51** (2004), 126–137, [nlin.SI/0311024](#).
- [12] Guo L., What is ... a Rota–Baxter algebra?, *Notices Amer. Math. Soc.* **56** (2009), 1436–1437.
- [13] Guo L., An introduction to Rota–Baxter algebra, *Surveys of Modern Mathematics*, Vol. 4, International Press, Somerville, MA, 2012.
- [14] Manakov S.V., Santini P.M., Inverse scattering problem for vector fields and the Cauchy problem for the heavenly equation, *Phys. Lett. A* **359** (2006), 613–619, [nlin.SI/0604024](#).
- [15] Manakov S.V., Santini P.M., A hierarchy of integrable partial differential equations in dimension $2 + 1$, associated with one-parameter families of vector fields, *Theoret. and Math. Phys.* **152** (2007), 1004–1011.
- [16] Manakov S.V., Santini P.M., On the solutions of the dKP equation: the nonlinear Riemann Hilbert problem, longtime behaviour, implicit solutions and wave breaking, *J. Phys. A: Math. Theor.* **41** (2008), 055204, 23 pages, [arXiv:0707.1802](#).
- [17] Manakov S.V., Santini P.M., The dispersionless 2D Toda equation: dressing, Cauchy problem, longtime behaviour, implicit solutions and wave breaking, *J. Phys. A: Math. Theor.* **42** (2009), 095203, 16 pages, [arXiv:0810.4676](#).
- [18] Manakov S.V., Santini P.M., On the solutions of the second heavenly and Pavlov equations, *J. Phys. A: Math. Theor.* **42** (2009), 404013, 11 pages, [arXiv:0812.3323](#).
- [19] Mañas M., On the r th dispersionless Toda hierarchy: factorization problem, additional symmetries and some solutions, *J. Phys. A: Math. Gen.* **37** (2004), 9195–9224, [nlin.SI/0404022](#).
- [20] Mañas M., S -functions, reductions and hodograph solutions of the r th dispersionless modified KP and Dym hierarchies, *J. Phys. A: Math. Gen.* **37** (2004), 11191–11221, [nlin.SI/0405028](#).
- [21] Martínez Alonso L., Shabat A.B., Energy-dependent potentials revisited: a universal hierarchy of hydrodynamic type, *Phys. Lett. A* **299** (2002), 359–365, [nlin.SI/0202008](#).

- [22] Martínez Alonso L., Shabat A.B., Towards a theory of differential constraints of a hydrodynamic hierarchy, *J. Nonlinear Math. Phys.* **10** (2003), 229–242, [nlin.SI/0310036](#).
- [23] Martínez Alonso L., Shabat A.B., Hydrodynamic reductions and solutions of the universal hierarchy, *Theoret. and Math. Phys.* **140** (2004), 1073–1085, [nlin.SI/0312043](#).
- [24] Pavlov M.V., Integrable hydrodynamic chains, *J. Math. Phys.* **44** (2003), 4134–4156, [nlin.SI/0301010](#).
- [25] Pavlov M.V., Chang J.-H., Chen Y.-T., Integrability of the Manakov–Santini hierarchy, [arXiv:0910.2400](#).
- [26] Rota G.C., Baxter algebras and combinatorial identities. I, *Bull. Amer. Math. Soc.* **75** (1969), 325–329.
- [27] Rota G.C., Baxter algebras and combinatorial identities. II, *Bull. Amer. Math. Soc.* **75** (1969), 330–334.
- [28] Semenov-Tian-Shansky M.A., Integrable systems and factorization problems, in Factorization and Integrable Systems (Faro, 2000), *Oper. Theory Adv. Appl.*, Vol. 141, Birkhäuser, Basel, 2003, 155–218, [nlin.SI/0209057](#).
- [29] Semenov-Tian-Shansky M.A., What is a classical r -matrix?, *Funct. Anal. Appl.* **17** (1983), 259–272.
- [30] Sergyeyev A., Szablikowski B.M., Central extensions of cotangent universal hierarchy: $(2 + 1)$ -dimensional bi-Hamiltonian systems, *Phys. Lett. A* **372** (2008), 7016–7023, [arXiv:0807.1294](#).
- [31] Szablikowski B.M., Classical r -matrix like approach to Frobenius manifolds, WDVV equations and flat metrics, *J. Phys. A: Math. Theor.* **48** (2015), 315203, 47 pages, [arXiv:1304.2075](#).
- [32] Szablikowski B.M., Błaszak M., Meromorphic Lax representations of $(1 + 1)$ -dimensional multi-Hamiltonian dispersionless systems, *J. Math. Phys.* **47** (2006), 092701, 23 pages, [nlin.SI/0510068](#).
- [33] Takasaki K., Takebe T., SDiff(2) KP hierarchy, *Internat. J. Modern Phys. A* **7** (1992), 889–922, [hep-th/9112046](#).
- [34] Takasaki K., Takebe T., Integrable hierarchies and dispersionless limit, *Rev. Math. Phys.* **7** (1995), 743–808, [hep-th/9405096](#).